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LINEAR EQUATIONS OF MOTION OF A CONCENTRATED DEFECT

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Abstract—The motion of a concentrated (point) defect in an elastic medium is investigated on the basis of a variational principle. The equations of motion and the principles of conservation of energy are derived and examined in some detail. The localization of the Lagrangian makes it possible to regularize its singular part and deduce explicit differential equations of motion. The radiation damping force is introduced by means of the Wheeler–Feynman procedure. In the paper we confine ourselves to the quadratic Lagrangian and hence linear equations of motion.

1. INTRODUCTION

IT is now a recognized fact that the motion of dislocations and other defects in a solid, significantly influences its properties and constitutes the basis for an explanation of various physical phenomena occurring in crystal structures. Consequently, it is necessary to formulate a general theory of motion of discrete defects; the motion of a discrete defect in a continuum is not only an interesting and important phenomenon in itself but should constitute the foundation for constructing a statistical theory of continuous distributions of defects;‡ the latter should justify (or introduce corrections to) the existing theory of continuous dislocations based on a number of postulates. The analogy can be drawn here with the classical hydrodynamics where the equations of motion can be derived either by means of purely phenomenological considerations (Cauchy laws) or by statistical methods on the basis of the (Newtonian or relativistic) particle mechanics; here, the phenomenological equations essentially depend on the equations of motion of a single particle and cannot be derived (and in fact were not derived) without a thorough knowledge of the properties of motion of the latter.

There have been very few attempts to derive the equations of motion of a defect; we mention here some papers having certain points in common with our treatment. J. D. Eshelby [1] was the first to derive in a rational way the equation of motion of a single dislocation. The problem was later investigated by A. M. Kosevitch [2]. The latter author used a variational principle; since, however, his general approach to the model of a defect in an elastic continuum and hence the Lagrangian, are essentially different from ours, the results are also significantly different. In [4] we made an attempt to construct the dynamics of defects in a linear isotropic elastic continuum on the basis of a variational principle, in the spirit of the general classical field theory, just as the classical electrodynamics or mesodynamics.

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‡ See [11] and [12].

In this paper we propose to investigate the self-force of a concentrated defect, defined in terms of the quantities characterizing the elastic field, in Section 2. It represents an idealized model of motion of a group of interstitial atoms or vacancies having the shape of a very small disc, or of a simplified very small Somigliana dislocation. The smallness of the surface of the defect makes it possible to derive exactly the equations of motion constituting a set of ordinary differential equations with constant coefficients, and the conservation principles. Thus, the motion can easily be investigated in all details by very simple mathematical methods.

We confine ourselves to linear expressions in velocities, i.e. to a quadratic Lagrangian. A localization (renormalization) of the latter leads to a definite explicit expression for the linear momentum of the dislocation, containing two (infinite) terms of the orders t_0^{-1} and t_0^{-3}, respectively, where t_0 is the time required for a sound signal to travel the distance equal to the diameter of the dislocation; the "mass" of the dislocation turns out to be a tensor of second order, the components of which in some cases may be negative. One of the above terms is proportional to the first and the second to the third derivative of the velocity of t' dislocation, the equations of motion therefore are of the fourth order. Further, by means of the Wheeler–Feynman procedure we derive an expression for the force due to the radiation damping; it is proportional to the fourth derivative of the velocity. The higher order of the derivatives as compared with, say, an electron in a Maxwell field, is due to the fact that we are dealing with double layer surface distributions [4]. Finally we write down the equations of motion and investigate their properties.

2. ACTION FUNCTIONAL AND ITS LOCALIZATION

Following the general idea of our earlier paper [4] we shall derive here the expression for the self-Lagrangian of a concentrated dislocation in a linear isotropic elastic continuum. The Lagrangian density constitutes a time integral and therefore the whole theory is non-local in time [cf. 9]; moreover, since the defect is concentrated, the above time integral in general does not exist. In order to obtain an explicit expression in terms of quantities at the instant t only, we shall employ a procedure which we call the localization of the Lagrangian density; here we follow the general method developed by P. G. Bergmann [3] for classical electrodynamics.

Consider therefore a very small moving surface $s(t)$ (Fig. 1) in an infinite classical elastic medium; the normal to the surface is denoted by $\mathbf{n}(t)$ and called the director of the

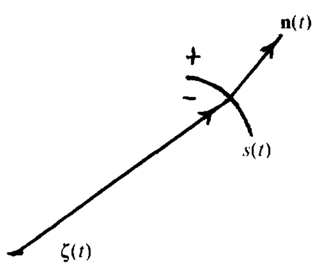

FIG. 1

defect; $\zeta(t)$ is its position vector. We assume that the displacement vector of the medium $\mathbf{u}(\mathbf{x}, t)$ suffers a discontinuity on $s(t)$, which is independent of time, i.e.

$$[\mathbf{u}(\zeta, t)] = \overset{+}{\mathbf{u}}(\zeta, t) - \overset{-}{\mathbf{u}}(\zeta, t) = -\mathbf{U}. \tag{2.1}$$

We assume in this paper that the director is also constant in time, i.e. $\dot{\mathbf{n}} = 0$. Since the surface $s(t)$ is very small, wherever convenient the integration over this surface can be replaced by a multiplication by the appropriate area, which in turn can be included into the definition of the discontinuity.

We are here interested in the self-Lagrangian only and therefore we omit the body forces or other external fields; thus, the total displacement $\mathbf{u}(\mathbf{x}, t)$ is due only to the defect. The action integral is taken in the form (see [4]).

$$W_T = W - \int_{t_1}^{t_2} \mathrm{d}t \int_{s(t)} \mathrm{d}a\, t_{(n)}\mathbf{u}(\zeta, t)[\mathbf{u}(\zeta, t)] + \frac{1}{2}\int_{t_1}^{t_2} \mathrm{d}t \int_{s(t)} \mathrm{d}a\, m\dot{\zeta}^2 \tag{2.2}$$

where

$$\begin{aligned} W &= W(\mathbf{u}, \mathbf{u}) = \int_{t_1}^{t_2} \mathrm{d}t \int_v \mathrm{d}v L\{\mathbf{u}, \mathbf{u}\} \\ L\{\mathbf{u}, \mathbf{u}\} &= T\{\mathbf{u}, \mathbf{u}\} - \Pi\{\mathbf{u}, \mathbf{u}\}. \end{aligned} \tag{2.3}$$

Here, m is the mass of the defect (if any) carried by its motion through the medium,

$$T\{\mathbf{u}, \mathbf{u}\} = \tfrac{1}{2}\rho\dot{\mathbf{u}}^2$$

is the kinetic energy density and

$$\Pi\{\mathbf{u}, \mathbf{u}\} = \tfrac{1}{2}[\lambda(\nabla . \mathbf{u})^2 + \tfrac{1}{2}\mu(\nabla\mathbf{u} + \mathbf{u}\nabla)^2]$$

is the potential energy density of the medium.

The first term in (2.2) is the ordinary action integral of classical elastic field, while the second represents the work done by the dynamic stress vector

$$t_{(n)}\mathbf{u} = \sigma_{(n)}\mathbf{u} + \rho v_{(n)}\dot{\mathbf{u}}$$

($\sigma_{(n)}\mathbf{u} = \mathbf{n} . \boldsymbol{\sigma}$ is the ordinary stress vector on s and $v_{(n)} = \mathbf{n} . \dot{\zeta} = \mathbf{n} . \mathbf{v}$ is the normal velocity of the dislocation) on the difference of the displacements of the two sides of the cut $s(t)$. Finally, the last integral in (2.2) is the ordinary kinetic energy of a moving mass; whether $m = 0$ or $m \neq 0$ is in a way immaterial, since this mass constitutes only a part of the total mass of the defect, the other part being the field mass; it is however convenient to keep this term in the calculations.

Integrating by parts we have

$$\begin{aligned} W\{\mathbf{u}, \mathbf{u}\} = &-\frac{1}{2}\int_{t_1}^{t_2} \mathrm{d}t \int_v \mathrm{d}v \mathbf{P}\mathbf{u} . \mathbf{u} + \frac{1}{2}\int_{t_1}^{t_2} \mathrm{d}t \int_{s(t)} \mathrm{d}a\, t_{(n)}\mathbf{u} . [\mathbf{u}] \\ &+ \tfrac{1}{2}\rho \int_v \mathrm{d}v\, \dot{\mathbf{u}} . \mathbf{u} \Big|_{t_1}^{t_2} \end{aligned} \tag{2.4}$$

where $\mathbf{P}$ is the Lamé operator. Taking into account that outside the singular surface

$$\mathbf{P}\mathbf{u} = 0$$

and using (2.1) we obtain dropping integration over $s(t)$

$$W_T = \frac{1}{2}\int_{t_1}^{t_2} \mathrm{d}t(m\dot{\zeta}^2 + t_{(n)}\mathbf{u} . \mathbf{U}) + \tfrac{1}{2}\rho \int_v \mathrm{d}v\dot{\mathbf{u}} . \mathbf{u} \Big|_{t_1}^{t_2} . \tag{2.5}$$

The last term is irrelevant, for in the variation we shall assume that $\delta\zeta(t_1) = \delta\zeta(t_2) = 0$; thus, setting $\mathbf{v} = \dot{\zeta}$ we finally have the following formula for the action integral of a defect in an elastic field:

$$W_T = \frac{1}{2}\int_{t_1}^{t_2} \mathrm{d}tm\mathbf{v}^2 + \frac{1}{2}\int_{t_1}^{t_2} \mathrm{d}t\mathbf{U} . t_{(n)}\mathbf{u}. \tag{2.6}$$

Obviously, the second term represents the field contribution to the action of the defect; here, $t_{(n)}\mathbf{u}(\zeta_t, t)$ is the dynamic stress vector due to the defect taken at the point of the defect at the instant t. Since the defect is concentrated, this expression is infinite. The localization procedure we now employ eliminates the time integral in $t_{(n)}\mathbf{u}$ and reduces the infinity of the whole expression to just infinite constants.

First, observe that the displacement $\mathbf{u}(\zeta_t, t)$ can be represented in the form [cf. 4, Section 2]

$$\mathbf{u}(\zeta, t) = -\int_{t_1}^{t_2} \mathrm{d}\tau\mathbf{U} . t_{(n)}\overset{+}{\mathbf{G}}(\zeta_\tau - \zeta_t, t-\tau) \tag{2.7}$$

where $\overset{+}{\mathbf{G}}$ is the Green tensor symmetric in time, equal to half the sum of the retarded and advanced Green tensors, i.e.

$$\overset{+}{\mathbf{G}} = \frac{1}{2}\left(\overset{\mathrm{ret}}{\mathbf{G}} + \overset{\mathrm{adv}}{\mathbf{G}}\right) \tag{2.8}$$

The displacement in (2.7) depends on both the past and future history of the defect, the final results however, after the localization, contain quantities at the instant t only. We shall return later to the problem of choosing the Green tensor in the expression (2.7) for the displacement.

Expanding the operator $t_{(n)}$ in (2.7), after simple transformations we obtain

$$u^i(\zeta, t) = -\int_{t_1}^{t_2} \mathrm{d}\tau(\mu U_j n_m \sigma^{jmpq}\nabla_p \overset{+}{G}{}^i_q + \rho v_{(n)} U_j \frac{\partial}{\partial\tau}\overset{+}{G}{}^{ji}) \tag{2.9}$$

where

$$\sigma^{jmpq} = \frac{\lambda}{\mu}\delta^{jm}\delta^{pq} + \delta^{jp}\delta^{mq} + \delta^{jq}\delta^{mp}. \tag{2.10}$$

Now, since

$$\begin{aligned}\int_{t_1}^{t_2} \mathrm{d}\tau v_{(n)} U_j \frac{\partial}{\partial\tau}\overset{+}{G}{}^{ji} = &-\frac{D}{Dt}\int_{t_1}^{t_2} \mathrm{d}\tau v_{(n)} U_j \overset{+}{G}{}^{ji} \\ &- v^p \int_{t_1}^{t_2} \mathrm{d}\tau v_{(n)} U_j \nabla_p \overset{+}{G}{}^{ji}\end{aligned} \tag{2.11}$$

we obtain

$$u^i = -\mu n_m U_j \sigma^{jmpq}\int_{t_1}^{t_2} d\tau \nabla_p \overset{+}{G}{}^i_q + \rho n_m U_j v^p \int_{t_1}^{t_2} d\tau v^m \nabla_p \overset{+}{G}{}^{ji} + \rho n_m U_j \frac{D}{Dt}\int_{t_1}^{t_2} d\tau v^m \overset{+}{G}{}^{ji}. \tag{2.12}$$

Let us now find the dynamic stress vector appearing in the action functional (2.6):

$$\begin{aligned} t_{(n)}u^i &= \mu n_n \sigma^{inrs}\nabla_r u_s + \rho v_{(n)}\frac{\partial}{\partial t}u^i \\ &= \mu n_n \sigma^{inrs}\nabla_r u_s + \rho v_{(n)}\left(\frac{D}{Dt}\, u^i - v^r \nabla_r u^i\right). \end{aligned} \tag{2.13}$$

Bearing in mind that $\dot{\mathbf{U}} = 0$ we can add to the above expression an arbitrary derivative with respect to time, without affecting the variation of the functional and therefore the equations of motion; thus, we may consider instead of (2.13) the expression

$$t_{(n)}u^i = \mu n_n \sigma^{inrs}\nabla_r u_s - \rho v_{(n)} v^r \nabla_r u^i - \rho n_n \dot{v}^n u^i. \tag{2.14}$$

Now, we substitute into the above formula the expression for $u^i(\zeta_t, t)$ and again neglect a derivative with respect to t; moreover, being interested only in the equations of motion linear with respect to the velocity of the defect or its derivatives, we neglect in the Lagrangian terms of an order higher than $\mathbf{vv}$. Thus, after some transformations we have

$$\begin{aligned} U_i t_{(n)}u^i = \mu\rho n_m n_n U_i U_j\bigg[& c_2^{-2}\ddot{v}^m \int_{t_1}^{t_2} d\tau v^n \overset{+}{G}{}^{ji} + \dot{v}^n \sigma^{jmpq}\int_{t_1}^{t_2} d\tau \nabla_p \overset{+}{G}{}^i_q \\ & -(v^n v^r \sigma^{jmpq}\delta^{is} - c_2^2 \sigma^{jmpq}\sigma^{inrs})\int_{t_1}^{t_2} d\tau \nabla_p \nabla_r \overset{+}{G}{}^i_q \\ & - v^p \sigma^{inrs}\int_{t_1}^{t_2} d\tau v^m \nabla_p \nabla_r \overset{+}{G}{}^{js}\bigg]. \end{aligned} \tag{2.15}$$

To calculate the (singular) integrals we first derive formulae for $\overset{+}{\mathbf{G}}$ and its derivatives. We have

$$\overset{\text{ret}}{G}{}^{ij}(\mathbf{r}, t-\tau) = \frac{1}{4\pi\rho}\left\{(t-\tau)\left[\eta\left(t-\tau-\frac{r}{c_1}\right) - \eta\left(t-\tau-\frac{r}{c_2}\right)\right]\partial^i\partial^j\frac{1}{r} + \frac{r^i r^j}{r^3}\left[\frac{\delta\left(t-\tau-\frac{r}{c_1}\right)}{c_1^2} - \frac{\delta\left(t-\tau-\frac{r}{c_2}\right)}{c_2^2}\right] + \frac{\delta^{ij}}{c_2^2}\,\frac{\delta\left(t-\tau-\frac{r}{c_2}\right)}{r}\right\}.$$

Taking into account that

$$t\left[\eta\left(t-\frac{r}{c_1}\right) - \eta\left(t-\frac{r}{c_2}\right)\right] = r^2\int_{c_2}^{c_1}\delta\left(t-\frac{r}{c}\right)\frac{dc}{c^3}$$

(this formula can readily be derived expressing η as an integral of δ) we obtain

$$\overset{\text{ret}}{G}{}^{ij}(\mathbf{r}, t-\tau) = \frac{1}{4\pi\rho}\left\{r^2\int_{c_2}^{c_1}\delta\left(t-\tau-\frac{r}{c}\right)\frac{\mathrm{d}c}{c^3}\partial^i\partial^j\frac{1}{r} + \frac{r^ir^j}{r^3}\left[\frac{\delta\left(t-\tau-\frac{r}{c_1}\right)}{c_1^2}-\frac{\delta\left(t-\tau-\frac{r}{c_2}\right)}{c_2^2}\right]+\frac{\delta^{ij}}{c_2^2}\frac{\delta\left(t-\tau-\frac{r}{c_2}\right)}{r}\right\}. \tag{2.16}$$

The physical sense of the first term is the following: it represents the signals collected from the part of the world line between the cones c_1 and c_2. For $\overset{\text{adv}}{\mathbf{G}}$ we have a similar formula, namely

$$\overset{\text{adv}}{G}{}^{ij}(\mathbf{r}, t-\tau) = \frac{1}{4\pi\rho}\left\{r^2\int_{c_2}^{c_1}\delta\left(t-\tau+\frac{r}{c}\right)\frac{\mathrm{d}c}{c^3}\partial^i\partial^j\frac{1}{r} + \frac{r^ir^j}{r^3}\left[\frac{\delta\left(t-\tau+\frac{r}{c_1}\right)}{c_1^2}-\frac{\delta\left(t-\tau+\frac{r}{c_2}\right)}{c_2^2}\right]+\frac{\delta^{ij}}{c_2^2}\frac{\delta\left(t-\tau+\frac{r}{c_2}\right)}{r}\right\}. \tag{2.17}$$

Making use of the formula

$$\frac{1}{2r}\left[\delta\left(t-\frac{r}{c}\right)+\delta\left(t+\frac{r}{c}\right)\right] = \frac{1}{c}\delta\left(t^2-\frac{r^2}{c^2}\right)$$

we finally arrive at the required expression for $\overset{+}{\mathbf{G}}$

$$\overset{+}{G}{}^{ij}(\mathbf{r}, t-\tau) = \frac{1}{4\pi\rho}\left\{r^3\partial^i\partial^j\frac{1}{r}\int_{c_2}^{c_1}\delta(\phi_c)\frac{\mathrm{d}c}{c^4} + \frac{r^ir^j}{r^2}\left[\frac{\delta(\phi_{c_1})}{c_1^3}-\frac{\delta(\phi_{c_2})}{c_2^3}\right]+\frac{\delta^{ij}}{c_2^3}\delta(\phi_{c_2})\right\} \tag{2.18}$$

where $(\tau-t=\theta)$

$$\phi_c = \phi_c(\theta) = \theta^2-\frac{1}{c^2}r^2(\theta).$$

The above form of $\overset{+}{\mathbf{G}}$ is however not entirely convenient, since in the derivatives of $\mathbf{G}$ there appear denominators containing r in high powers; consequently we integrate by parts the first term in (2.18) and we have†

$$\overset{+}{G}{}^{ij}(\mathbf{r}, \theta) = \frac{1}{4\pi\rho}\left\{\int_{c_2}^{c_1}\left[-\delta^{ij}\delta(\phi_c)+\frac{2r^ir^j}{c^2}\delta'(\phi_c)\right]\frac{\mathrm{d}c}{c^4}+\frac{\delta^{ij}}{c_2^3}\delta(\phi_{c_2})\right\} \tag{2.19}$$

† Sometimes the following form of $\overset{+}{\mathbf{G}}$ is useful:

$$\overset{+}{G}{}^{ij}(\mathbf{r}, \theta) = \frac{1}{4\pi\rho}\left[\frac{1}{2}\nabla^i\nabla^j\int_{c_2}^{c_1}\eta(\phi_c)\frac{\mathrm{d}c}{c^2}+\frac{\delta^{ij}}{c_2^3}\delta(\phi_{c_2})\right]$$

The retarded and advanced Green tensors can also be represented in a similar form, namely

$$\overset{\text{ret}}{G}{}^{ij}(\mathbf{r}, t-\tau) = \frac{1}{4\pi\rho}\frac{1}{r}\left\{\int_{c_2}^{c_1}\frac{\mathrm{d}c}{c^3}\left\{-\delta^{ij}\delta\left(t-\tau-\frac{r}{c}\right)+\frac{r^ir^j}{r^2}\frac{\partial}{\partial c}\left[c\delta\left(t-\tau-\frac{r}{c}\right)\right]\right\}+\frac{\delta^{ij}}{c_2^2}\delta\left(t-\tau-\frac{r}{c_2}\right)\right\}$$

and similarly for $\overset{\text{adv}}{\mathbf{G}}{}^{ij}(r, t-\tau)$.

Differentiating the above expression we obtain

$$\nabla^p \overset{+}{G}{}^{ij} = \frac{1}{2\pi\rho}\left\{\int_{c_2}^{c_1}\frac{dc}{c^6}\left[3\delta^{(ij}r^{p)}\delta'(\phi_c)-\frac{2}{c^2}r^ir^jr^p\delta''(\phi_c)\right]\right.$$
$$\left.-\frac{\delta^{ij}}{c_2^5}r^p\delta'(\phi_{c_2})\right\} \tag{2.20}$$
$$\nabla^p\nabla^q \overset{+}{G}{}^{ij} = \frac{1}{2\pi\rho}\left\{\int_{c_2}^{c_1}\frac{dc}{c^6}\left[3\delta^{(ij}\delta^{p)q}\delta'(\phi_c)-\frac{12}{c^2}\delta^{(ij}r^pr^{q)}\delta''(\phi_c)\right.\right.$$
$$\left.\left.+\frac{4}{c^4}r^ir^jr^pr^q\delta'''(\phi_c)\right]-\frac{\delta^{ij}}{c_2^5}\left[\delta^{pq}\delta'(\phi_{c_2})-\frac{2}{c_2^2}r^pr^q\delta''(\phi_c)\right]\right\}.$$

Observe that we are outside the scope of the ordinary theory of generalized functions, for $\delta(\phi_c)$ and its derivatives are multiplied by functions which are not sufficiently regular. Consequently, not all operations admissible in this theory are admissible in our case; for instance integration by parts is not allowed.

Consider now the generalized functions $\delta(\phi_c)$, $\delta'(\phi_c)$, etc., appearing in the above formulae. We have $\phi_c\delta(\phi_c) = 0$; differentiating this relation with respect to ϕ_c we obtain

$$\phi_c\delta'(\phi_c)+\delta(\phi_c) = 0, \text{ i.e. } \phi_c\delta'(\phi_c) = -\delta(\phi_c)$$
$$\phi_c\delta''(\phi_c)+2\delta'(\phi_c) = 0, \text{ i.e. } \phi_c^2\delta''(\phi_c) = 2\delta(\phi_c) \tag{2.21}$$
$$\phi_c\delta'''(\phi_c)+3\delta''(\phi_c) = 0, \text{ i.e. } \phi_c^3\delta'''(\phi_c) = -6\delta(\phi_c).$$

Multiplying now in (2.19) and (2.20) $\delta'(\phi_c)$ and its derivatives by suitable powers of ϕ_c and applying the formulae (2.21) we obtain

$$\overset{+}{G}{}^{ij}(\mathbf{r},\theta) = \frac{1}{4\pi\rho}\left\{-\int_{c_2}^{c_1}\frac{dc}{c^4}\left[\delta^{ij}\delta(\phi_c)+2\underline{\frac{r^ir^j}{c^2}\frac{\delta(\phi_c)}{\phi_c}}\right]+\frac{\delta^{ij}}{c_2^3}\delta(\phi_c)_2\right\}$$

$$\nabla^p\overset{+}{G}{}^{ij}(\mathbf{r},\theta) = \frac{1}{2\pi\rho}\left\{-\int_{c_2}^{c_1}\frac{dc}{c^6}\left(3\delta^{(ij}r^{p)}\frac{\delta(\phi_c)}{\phi_c}+\underline{\frac{4}{c^2}r^ir^jr^p\frac{\delta(\phi_c)}{\phi_c^2}}\right)\right.$$
$$\left.+\frac{\delta^{ij}}{c_2^5}r^p\frac{\delta(\phi_{c_2})}{\phi_{c_2}}\right\} \tag{2.22}$$
$$\nabla^p\nabla^q\overset{+}{G}{}^{ij}(\mathbf{r},\theta) = \frac{1}{2\pi\rho}\left\{-\int_{c_2}^{c_1}\frac{dc}{c^6}\left[3\delta^{(ij}\delta^{p)q}\frac{\delta(\phi_c)}{\phi_c}\right.\right.$$
$$\left.+\frac{24}{c^2}\delta^{(ij}r^pr^{q)}\frac{\delta(\phi_c)}{\phi_c^2}+\underline{\frac{24}{c^4}r^ir^jr^pr^q\frac{\delta(\phi_c)}{\phi_c^3}}\right]$$
$$\left.+\frac{\delta^{ij}}{c_2^5}\left[\delta^{pq}\frac{\delta(\phi_{c_2})}{\phi_{c_2}}+\frac{4}{c_2^2}r^pr^q\frac{\delta(\phi_{c_2})}{\phi_{c_2}^2}\right]\right\}.$$

Bearing in mind the expression (2.15) we observe that the underlined terms in (2.22) lead to powers higher than the second in $\mathbf{v}$; therefore, they can be omitted in calculating the integrals.

Consider now a typical integral appearing in (2.15). We replace integration over τ by integration over θ; thus, we shall examine the integral

$$c_1 = \int d\theta \Psi(\theta)\delta(\phi_c)$$

where $\Psi(\theta)$ is a function analytic in the vicinity of $\theta = 0$, i.e.

$$\Psi(\theta) = \Psi_0 + \theta\Psi_1 + \theta^2\Psi_2 + \ldots$$

To calculate the integral we introduce a new variable

$$\theta^2 - \frac{1}{c^2} r^2(\theta) = \phi_c = z^2.$$

Now, on the world line of the defect†

$$r^i(\theta) = \theta\left(v^i + \frac{\theta}{2}\dot{v}^i + \frac{\theta^2}{6}\ddot{v}^i + \ldots\right) \tag{2.23}$$

hence

$$r^2(\theta) = \theta^2\left[v^2 + \theta v\dot{v} + \theta^2\left(\frac{\dot{v}^2}{4} + \frac{v\ddot{v}}{3}\right) + \ldots\right] \tag{2.24}$$

consequently,

$$\theta = z(\alpha_c + \beta_c z + \gamma_c z^2 + \ldots), \qquad d\theta = (\alpha_c + 2\beta_c z + 3\gamma_c z^2 + \ldots)\, dz \tag{2.25}$$

where

$$\alpha_c = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}, \qquad \beta_c = \frac{v\dot{v}}{2c^2}\left(1 - \frac{v^2}{c^2}\right)^{-2},$$

$$\gamma_c = \frac{5}{8}\frac{(v\dot{v})^2}{c^4}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{7}{2}} + \frac{1}{2c^2}\left(\frac{\dot{v}^2}{4} + \frac{v\ddot{v}}{3}\right)\left(1 - \frac{v^2}{c^2}\right)^{-\frac{5}{2}}.$$

It can readily be verified that the terms omitted in the above expressions drop out, for

$$\int \delta(z^2) z^{2n}\, dz = 0 \quad \text{for } n \geq 1$$

and, in view of the antisymmetry,

$$\int \delta(z^2) z^{2n+1}\, dz = 0 \quad \text{for } n \geq 0.$$

Introduce now the notations

$$\Delta = \int \delta(z^2)\, dz, \qquad \Delta^1 = \int \frac{\delta(z^2)}{z^2}\, dz, \qquad \Delta^2 = \int \frac{\delta(z^2)}{z^4}\, dz. \tag{2.26}$$

In the sense of any existing theory of generalized functions these integrals do not exist or, which in our case is equivalent, remain undefined; a change of the integration variable leads to the formulae

$$\Delta = \frac{1}{2}\int \frac{\delta(z')}{z'^{\frac{1}{2}}}\, dz', \qquad \Delta^1 = \frac{1}{2}\int \frac{\delta(z')}{z'^{\frac{3}{2}}}\, dz', \qquad \Delta^2 = \frac{1}{2}\int \frac{\delta(z')}{z'^{\frac{5}{2}}}\, dz'. \tag{2.27}$$

† Observe that we approach the singularity of the defect along the world-line. The final results do depend on the manner of passing to the limit.

The first of the above integrals was used by Bergmann [3]. All integrals are infinite owing to the presence of a singularity in the Green tensor and the fact that the dislocation is infinitesimal; if it were finite the considered integrals would also be finite; on this basis we can estimate their magnitude. In fact, observe that z has the dimension of time and if t_0 is the time required for a sound signal to travel a distance equal to the diameter of the dislocation, then we have

$$\Delta = t_0^{-1}\Delta^*, \qquad \Delta^1 = t_0^{-3}\Delta^{1*}, \qquad \Delta^2 = t_0^{-5}\Delta^{2*}$$

where Δ^*, Δ^{1*}, Δ^{2*} are finite undetermined constants.

Making use of the formulae (2.23)–(2.25) and omitting terms leading to powers higher than v^2 it is easy to prove that

$$\int d\theta\Psi(\theta)\delta(\phi_c) = \alpha\Psi_0\Delta$$

$$\int d\theta\Psi(\theta)\frac{\delta(\phi_c)}{\phi_c} = \alpha\Psi_0\Delta^1 + (3\gamma\Psi_0 + 3\alpha\beta\Psi_1 + \alpha^3\Psi_2)\Delta$$

$$\int d\theta\Psi(\theta)\theta^2\frac{\delta(\phi_c)}{\phi_c^2} = \alpha^3\Psi_0\Delta^2 + (5\gamma\Psi_0 + 5\alpha\beta\Psi_1 + \alpha^5\Psi_2)\Delta^1.$$

We are now in a position to compute the integrals in (2.15); thus, retaining only the terms relevant in our quadratic Lagrangian we have

$$\int d\theta\ddot{v}^n\overset{+}{G}{}^{ji} = \frac{\delta^{ij}}{12\pi\rho}(c_1^{-3} + 2c_2^{-3})\ddot{v}^n\Delta + O(v^2)$$

$$\int d\theta\nabla_p\overset{+}{G}{}^{qi} = \frac{1}{4\pi\rho}[\tfrac{3}{5}(c_1^{-5} - c_2^{-5})\delta^{(qi}\dot{v}^{p)} + \delta^{qi}c_2^{-5}\dot{v}^p]\Delta + O(v^2)$$

$$\int d\theta\nabla_p\nabla_r\overset{+}{G}{}^{qs} = \frac{1}{2\pi\rho}[\tfrac{3}{5}(c_1^{-5} - c_2^{-5})\delta^{(qs}\delta^{p)r} + c_2^{-5}\delta^{qs}\delta^{pr}]\Delta^1 + O(v)$$

$$\int d\theta\nabla_p\nabla_r\overset{+}{G}{}^{qs} = \frac{1}{14\pi\rho}v^kv^l\alpha_{kl}^{qspr}\Delta^1 + \dot{v}^k\dot{v}^l\beta_{kl}^{qspr}\Delta) + O(v^3)$$

$$+ \text{ terms independent of } v + \text{ terms linear in } v$$

where

$$\alpha_{kl}^{qspr} = a^{qspr}\delta_{kl} + b_{kl}^{qspr}\beta_{kl}^{qspr} = -\tfrac{1}{4}(a^{qspr}\delta_{kl} + \tfrac{1}{3}b_{kl}^{qspr})$$

and

$$a^{qspr} = \tfrac{1}{2}[3(c_1^{-7} - c_2^{-7})\delta^{(qs}\delta^{p)r} + c_2^{-7}\delta^{qs}\delta^{pr}]$$

$$b_{kl}^{qspr} = 4[6(c_1^{-7} - c_2^{-7})\delta^{(qs}\delta_k^p\delta_1^{r)} + 7c_2^{-7}\delta^{qs}\delta_k^p\delta_1^r]$$

Now we can easily calculate the Lagrangian density $\mathbf{U}\,.\,t_{(n)}\mathbf{u}$; taking into account that the expression $v^i\ddot{v}^j$ may be replaced by $-\dot{v}^i\dot{v}^j$ by adding a time derivative, after simple though cumbersome transformations we obtain

$$\mathbf{U}\,.\,t_{(n)}\mathbf{u} = m^{pq}v_pv_q + n^{pq}\dot{v}_p\dot{v}_q \tag{2.28}$$

where

$$m^{pq} = \mu c_2^{-5}\Delta^1[\delta^{pq}(m_1 U^2 + m_2 U_{(n)}^2) + m_3 U_{(n)} n_{(p} U_{q)} + m_4 U_p U_q + m_5 U_2 n_p n_q]$$
$$n_{pq} = \mu c_2^{-5}\Delta[\delta^{pq}(n_1 U^2 + n_2 U_{(n)}^2) + n_3 U_{(n)} n_{(p} U_{q)} + n_4 U_p U_q + n_5 U^2 n_p n_q] \tag{2.29}$$

the coefficients $m_1, \ldots, n_5$ are dimensionless and depend only on the ratio of the sound velocities $s = c_2/c_1$; this quantity varies from zero (for $\nu = \frac{1}{2}$) to $1/\sqrt{2}$ (for $\nu = 0$). We have

$$m_1 = \frac{1}{14\pi}(2s^7+5), \qquad m_2 = \frac{1}{28\pi}(284s^7 - 328s^5 + 95s^3 - 18 + 48s^{-2} - 18s^{-4}),$$
$$m_3 = \frac{1}{14\pi}\left(-160s^7 + \frac{644}{5}s^5 - 14s^3 - \frac{54}{5}\right), \qquad m_4 = \frac{2}{7\pi}(4s^7+3),$$
$$m_5 = \frac{1}{14\pi}\left(16s^7 - \frac{28}{5}s^5 + \frac{18}{5}\right)$$
$$n_1 = \frac{1}{56\pi}(2s^7+5), \qquad n_2 = -\frac{1}{56\pi}\left(\frac{170}{3}s^7 - \frac{204}{3}s^5 + \frac{125}{6}s^3 + \frac{5}{3} + \frac{8}{3}s^{-2} - s^{-4}\right) \tag{2.30}$$
$$n_3 = \frac{1}{2\pi}\left(\frac{40}{21}s^7 - \frac{38}{15}s^5 + s^3 + \frac{31}{105}\right), \qquad n_4 = -\frac{1}{42\pi}(4s^7+3)$$
$$n_5 = \frac{1}{2\pi}\left(-\frac{4}{21}s^7 + \frac{2}{3}s^5 - \frac{1}{6}s^3 + \frac{13}{105}\right).$$

Observe that the components of the tensors m^{pq} and n^{pq} can be both positive and negative, depending on the type of the defect and the properties of the elastic medium—its Poisson ratio ν, i.e. the parameter s. In fact, in the case of a normal defect, i.e. one for which $U_\Delta = 0$ for small s the dominant term (arising from m_2) is $-(9/14\pi)\mu c_2^{-5}\Delta^1 v^2 U_{(n)}^2 s^{-4}$ in the first term of the Lagrangian; analogous result holds for the second term. However, in the case of a tangential defect we always have

$$m^{pq} > 0$$
$$n^{pq} < 0 \tag{2.31}$$

It is of course important to establish the sign of the quadratic forms $m^{pq}v_p v_q$ and $n^{pq}\dot{v}_p\dot{v}_q$. Let us examine successively the normal and tangential defects.

In the case of a normal defect, without affecting the generality we may set

$$n^i = (1,0,0), \qquad u_i = u(1,0,0).$$

Then

$$m_{11} = m_1 + m_2 + m_3 + m_4 + m_5$$
$$= \mu c_2^{-5}\Delta^1 u^2 \frac{1}{140\pi}(160s^7 - 408s^5 + 335s^3 + 8 + 240s^{-2} - 90s^{-4})$$

$$m_{22} = m_{33} = m_1 + m_2 = \mu c_2^{-5}\Delta^1 u^2 \frac{1}{28\pi}(288s^7 - 328s^5 + 95s^3 + 8 + 48s^{-2} - 18s^{-4})$$

$$m_{12} = m_{13} = m_{23} = 0.$$

The polynomial appearing in m_{11} changes its sign in the considered range of $s(s = 0{,}5906$ is its root) and hence the quadratic form $m^{pq}v_p v_q$ is indefinite and its sign depends on the Poisson number ν. The same concerns the form $n^{pq}\dot{v}_p\dot{v}_q$.

The case is essentially different when the defect is tangential. Here we set

$$n_i = (1, 0, 0), \qquad U_i = u(0, 1, 0)$$

and then

$$m_{11} = m_1 + m_5 = \frac{1}{14\pi}\left(18s^7 - \frac{28}{5}s^5 + \frac{43}{5}\right) > 0,$$

$$m_{22} = m_1 + m_4 = \frac{1}{146}(18s^7 + 17) > 0,$$

$$m_{33} = m_1 = \frac{1}{14\pi}(2s^7 + 5) > 0, \qquad m_{12} = m_{13} = m_{23} = 0$$

$$n_{11} = n_1 + n_5 = \frac{1}{2\pi}\left[-\frac{11}{48}s^7 + \frac{2}{5}s^5 - \frac{1}{6}s^3 + \left(-\frac{5}{28} + \frac{13}{105}\right)\right] < 0$$

$$n_{22} = n_1 + n_4 = -\frac{1}{14\pi}\left[\left(\frac{1}{2} + \frac{4}{3}\right)s^7 + \left(\frac{5}{4} + 1\right)\right] < 0,$$

$$n_{33} = n_1 = -\frac{1}{56\pi}(2s^7 + 5) < 0, \qquad n_{12} = n_{13} = n_{23} = 0.$$

Thus, applying simple criteria of definiteness of quadratic forms we obtain the following important result:

$$m^{pq}v_p v_q > 0, \qquad n^{pq}\dot{v}_p\dot{v}_q < 0. \tag{2.32}$$

Let us now return to the representation (2.9), of the displacement produced by the dislocation, in terms of the Green tensor. We have been using so far the symmetric tensor $\overset{+}{\mathbf{G}}$ which yields the Lagrangian density invariant with respect to the change of the time direction and makes it possible to formulate in the non-local theory the variational principle; since, however, we localize the Lagrangian density, the last merit of the Green tensor $\overset{+}{\mathbf{G}}$ is irrelevant and there are no principal objections against using the retarded tensor $\overset{\text{ret}}{\mathbf{G}}$. If we introduce $\overset{\text{ret}}{\mathbf{G}}$ instead of $\overset{+}{\mathbf{G}}$, however, it turns out that no new terms appear in the Lagrangian. We shall prove this statement. It is convenient to write

$$\overset{\text{ret}}{\mathbf{G}} = \overset{+}{\mathbf{G}} + \overset{-}{\mathbf{G}} \tag{2.33}$$

where $\overset{-}{\mathbf{G}} = 1/2(\overset{\text{ret}}{\mathbf{G}} - \overset{\text{adv}}{\mathbf{G}})$; hence (we use the notation $\mathbf{u} = \overset{+}{\mathbf{u}}$).

$$\overset{\text{ret}}{\mathbf{u}} = \overset{+}{\mathbf{u}} + \overset{-}{\mathbf{u}} \tag{2.34}$$

and

$$\mathbf{U} \,.\, t_{(n)}\overset{\text{ret}}{\mathbf{u}} = \mathbf{U} \,.\, t_{(n)}\mathbf{u} + \mathbf{U} \,.\, t_{(n)}\overset{-}{\mathbf{u}}. \tag{2.35}$$

Simple calculations carried out as for the tensor $\overset{+}{\mathbf{G}}$ or by expanding $\mathbf{u}$ into the series of instantaneous potentials, lead to the following result:

$$\mathbf{U} \,.\, t_{(n)}\bar{\mathbf{u}} = 0, \qquad \text{i.e. } \mathbf{U} \,.\, t_{(n)}\overset{\text{ret}}{\mathbf{u}} = \mathbf{U} \,.\, t_{(n)}\mathbf{u} \tag{2.36}$$

it is therefore irrelevant whether, at least in the quadratic approximation of the Lagrangian, we use $\overset{+}{\mathbf{G}}$ or $\overset{\text{ret}}{\mathbf{G}}$ in our localization. It is interesting to note that in the expression $t_{(n)}\mathbf{u} = \sigma_{(n)}\mathbf{u} + \rho v_{(n)}\dot{\mathbf{u}}$ each of the two terms contributes to the expression for the Lagrangian density, these contributions being the following:

$$\frac{\rho}{60\pi c_2^3}(-6s^5+5s^3+1)U_{(n)}\dot{v}_{(n)}\ddot{v}_p u^p \quad \text{and} \quad -\frac{\rho}{60\pi c_2^3}(-6s^5+5s^3+1) \tag{2.37}$$

$U_{(n)}\dot{v}_{(n)}\ddot{v}_p u^p$, respectively; thus, only their sum vanishes.

We are now in a position to write the action functional in the form

$$W = \int_{t_1}^{t_2} \mathrm{d}t L(\mathbf{v}, \dot{\mathbf{v}}) \tag{2.38}$$

where

$$L(\mathbf{v}, \dot{\mathbf{v}}) = \tfrac{1}{2}(m^{*pq}v_p v_q + n^{pq}\dot{v}_p\dot{v}_q) \tag{2.39}$$

and $m^{*pq} = m\delta^{pq} + m^{pq}$. On the basis of (2.39) we can of course construct the Hamiltonian and the whole Hamilton formalism. We confine ourselves here to deriving the Hamiltonian only. Thus, we have [see 10],

$$H = -L + v^p p_p + \dot{v}^p r_p \tag{2.40}$$

where the generalized momentum is given by the formula

$$p_i = \frac{\partial L}{\partial v^i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{v}^i}\right) \tag{2.41}$$

and

$$r_i = \frac{\partial L}{\partial \dot{v}^i}.$$

Making use of (2.39) we obtain the required expression

$$H = \tfrac{1}{2}(m^{*pq}v_p v_q + n^{pq}\dot{v}_p\dot{v}_q) - n^{pq}v_p\ddot{v}_q = L - n^{pq}v_p\ddot{v}_q. \tag{2.42}$$

It is easy to guess that the above Hamiltonian is conserved during the motion of the dislocation and will be identified with its energy (cf. Section 3).

3. EQUATIONS OF MOTION AND THEIR PROPERTIES—CONSERVATION OF ENERGY

Now we can derive the equations of motion and the principle of conservation of energy. We postulate the variational principle

$$\delta W = 0 \tag{3.1}$$

where

$$W\{\zeta(t)\} = \int_{t_1}^{t_2} \mathrm{d}t L[\zeta(t), \dot{\zeta}(t), \ddot{\zeta}(t)]. \tag{3.2}$$

The Lagrangian density L does not contain in our case $\zeta(t)$, the notation (3.2), however, is convenient in our further considerations. The variation δ in the principle (3.1) is the total variation of the functional, due to the variation of the form of the function $\zeta(t)$ and the variation of the independent variable t; we denote

$$\delta t = \varepsilon, \qquad \delta\zeta_i = \xi_i\varepsilon \tag{3.3}$$

then the variation of $\zeta(t)$ produced by the change of the form of the function only, is given by the formula

$$\bar{\delta}\zeta_i = (\xi_i - \dot{\zeta}_i)\varepsilon. \tag{3.4}$$

The total variation (3.1) can be written in the form of the sum

$$\delta W = \int_{t_1}^{t_2} \mathrm{d}t\, \delta L + \int_{t_1}^{t_2} \delta(\mathrm{d}t) L. \tag{3.5}$$

Taking into account that

$$\delta(\mathrm{d}t) = \mathrm{d}t\frac{\mathrm{d}}{\mathrm{d}t}(\delta t), \qquad \delta L = \bar{\delta}L + \frac{\mathrm{d}L}{\mathrm{d}t}\delta t \tag{3.6}$$

where

$$\bar{\delta}L = \frac{\partial L}{\partial\zeta_p}\bar{\delta}\zeta_p + \frac{\partial L}{\partial\dot{\zeta}_p}\frac{\mathrm{d}}{\mathrm{d}t}(\bar{\delta}\zeta_p) + \frac{\partial L}{\partial\ddot{\zeta}_p}\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\bar{\delta}\zeta_p) \tag{3.7}$$

(the variation $\bar{\delta}$ commutes with the differentiation with respect to time) and substituting into (3.5) we obtain the total variation of the considered functional

$$\delta W = \int_{t_1}^{t_2} \mathrm{d}t \left\{ \left[\frac{\partial L}{\partial\zeta_p}\bar{\delta}\zeta_p + \frac{\partial L}{\partial\dot{\zeta}_p}\frac{\mathrm{d}}{\mathrm{d}t}(\bar{\delta}\zeta_p) + \frac{\partial L}{\partial\ddot{\zeta}_p}\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\bar{\delta}\zeta_p) \right] + \frac{\mathrm{d}}{\mathrm{d}t}(L\delta t) \right\}$$

after simple transformations of the square brackets, making use of (3.4) and (3.3) we have finally

$$\delta W = \int_{t_1}^{t_2} \mathrm{d}t [L]^p \bar{\delta}\zeta_p - \int_{t_1}^{t_2} \mathrm{d}t \frac{\mathrm{d}\theta}{\mathrm{d}t}\varepsilon \tag{3.8}$$

where $[L]^i$ is the Lagrangian derivative

$$[L]^i = \frac{\partial L}{\partial\zeta_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\zeta}_i}\right) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial L}{\partial\ddot{\zeta}_i}\right) \tag{3.9}$$

and

$$\theta = -\left\{ \left[\frac{\partial L}{\partial\dot{\zeta}_p} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\ddot{\zeta}_p}\right) \right](\zeta_p - \dot{\zeta}_p) + \frac{\partial L}{\partial\ddot{\zeta}_p}(\dot{\zeta}_p - \ddot{\zeta}_p) + L \right\}. \tag{3.10}$$

Thus, the variational principle (3.1) and the du Bois–Reymond lemma lead to the following form of the Noether theorem [6]: in order that the functional W be invariant with respect to the transformation (3.3) it is necessary and sufficient that for an arbitrary t in the interval (t_1, t_2),

$$[L]^p \bar{\delta}\zeta_p - \frac{\mathrm{d}\theta}{\mathrm{d}t}\varepsilon = 0. \tag{3.11}$$

If we, moreover, require that the Euler equations of the problem be satisfied,

$$[L]^i = 0, \qquad \text{i.e. } n^{pq}\dddot{v}_q - m^{*pq}\dot{v}_q = 0 \tag{3.12}$$

the relation (3.11) yields the principle of conservation of the quantity $\theta(t)$, namely

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0. \tag{3.13}$$

Equation (3.12) are the required equations of motion of a concentrated defect in an elastic field, when no external forces are present.

Assume now that the transformation (3.3) is a shift of the time axis; then

$$\delta\zeta_i = 0 \qquad \text{i.e. } \xi_i = 0 \tag{3.14}$$

and for the quantity $\theta(t)$ we obtain the formula (cf. 2.42)

$$\theta = \tfrac{1}{2}(m^{*pq}v_p v_q + n^{pq}\dot{v}_p\dot{v}_q) - n^{pq}v_p\ddot{v}_q = H. \tag{3.15}$$

This quantity will be called the energy of the defect and the law (3.13) the principle of conservation of energy. In contrast to the kinetic energy of a Newtonian particle, the energy given by (3.15) is not positive definite. This energy is positive definite if and only if the defect is tangential and the motion is uniform (cf. 2.31).

Evidently, the principle of conservation of energy can be derived directly from the equations of motion. If there exists an external force acting on the defect, the equations of motion take the form

$$n^{pq}\ddot{v}_q - m^{*pq}\dot{v}_q = -f^p. \tag{3.16}$$

Multiplying (3.16) by v^p, after simple transformations we obtain

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \mathbf{f}\,.\,\mathbf{v} \tag{3.17}$$

which in the case $\mathbf{f} = 0$ yields (3.13). The law (3.17) states simply that the change of energy of the defect is equal to the work done by the external force.

We now proceed to investigate the properties of the equations of motion; the latter can be written in the form

$$n^{pq}\ddddot{\zeta}_q - m^{*pq}\ddot{\zeta}_q = 0. \tag{3.18}$$

This is a system of three ordinary differential equations with the unknown function $\zeta(t)$. One of its important properties is the fact that in the general case the equations are coupled, in view of the tensorial nature of the mass of the dislocation m^{*pq} and the field mass of second kind n^{pq}. It follows from this property that in general there does not exist a motion in an arbitrary constant direction, which we denote by v^i. In fact, then we have

$$v^i = v\nu^i, \qquad \dot{\nu}^i = 0, \qquad |\nu^i| = 1 \tag{3.19}$$

and hence, the equations of motion take the form

$$n^{pq}\nu_q\ddot{v} - m^{*pq}\nu_q\dot{v} = 0. \tag{3.20}$$

This is a system of three equations with one unknown function $v(t)$; assuming that $\dot{v}(t) \neq 0$, in order that the above equations be reduced to one, we must have

$$n^{pq}\nu_q = cm^{*pq}\nu_q \tag{3.21}$$

where c is an arbitrary constant; for an arbitrary v^i this condition is satisfied if and only if $n^{pq} = cm^{*pq}$ which in accordance with (2.29) and (2.30) never takes place for $U \neq 0$. Thus, a defect set in motion either moves uniformly along a straight line, or it moves along a curved trajectory. By setting the defect into motion we understand prescribing for $t = 0$ the four quantities $\zeta(0)$, $v(0)$, $\dot{v}(0)$ and $\ddot{v}(0)$. The case is entirely different when the direction of the initial motion is not arbitrary but connected with the structure of the defect. Set for instance $v^i = n^i$; then, instead of the condition (3.21) we have

$$n^{pq}n_q = cm^{pq}n_q \tag{3.22}$$

i.e., according to (2.29)

$$\begin{aligned} n^p[(n_1+n_5)U^2+(n_2+\tfrac{1}{2}n_3)U^2_{(n)}]+U^p(\tfrac{1}{2}n_3+n_4)U_{(n)} \\ = c\frac{\Delta^1}{\Delta}\left\{n^p\left[\frac{mc_2^5}{\mu\Delta^1}+(m_1+m_5)U^2+(m_2+\tfrac{1}{2}m_3)U^2_{(n)}\right]\right. \\ \left.+U^p(\tfrac{1}{2}m_3+m_4)U_{(n)}\right\}. \end{aligned} \tag{3.23}$$

With an appropriate choice of c this condition is satisfied in the following two cases:

(i) $U_{(n)} = 0$ (Tangential defect),

$$c = \frac{\Delta}{\Delta^1}\frac{(n_1+n_5)U^2}{(mc_2^5/\mu\Delta^1)+(m_1+m_5)U^2} \tag{3.24}$$

(ii) $U^i = Un^i$ (Normal defect),

$$c = \frac{\Delta}{\Delta^1}\frac{(n_1+n_2+n_3+n_4+n_5)U^2}{(mc_2^5/\mu\Delta^1)+(m_1+m_2+m_3+m_4+m_5)U^2}.$$

In these cases we are faced with non-uniform motion along the straight line defined by the director n^i.

Suppose now that $v^i = t^i$, $\mathbf{t}.\mathbf{n} = 0$; now we have

$$\begin{aligned} t^p(n_1U^2+n_2U^2_{(n)})+\tfrac{1}{2}n^pn_2U_{(n)}U_{(t)}+U^pn_4U_{(t)} \\ = c\frac{\Delta^1}{\Delta}\left[t^p\left(\frac{mc_2^5}{\mu\Delta^1}+m_1U^2+m_2U^2_{(n)}\right)+\tfrac{1}{2}n^pm_3U_{(n)}U_{(t)}+U^pm_4U_{(t)}\right] \end{aligned} \tag{3.25}$$

Now, too, the condition is satisfied for both normal and tangential defects. Thus, we have

(i) $U_{(n)} = 0$,

$$c = \frac{\Delta}{\Delta^1}\frac{(n_1+n_4)U^2}{(mc_2^5/\mu\Delta^1)+(m_1+m_4)U^2}$$

(ii) $U^i = Un^i$

$$c = \frac{\Delta}{\Delta^1}\frac{(n_1+n_2)U^2}{(mc_2^5/\mu\Delta^1)+(m_1+m_2)U^2} \tag{3.26}$$

and a non-uniform motion in the constant direction t^i is possible.

In all above cases the equation of motion has the form

$$c\ddddot{v}-\dot{v} = 0 \tag{3.27}$$

where in accordance with (3.24) and (3.26) c is a small constant parameter of the order t_0^2. It is of course important to establish the sign of c. A simple calculation shows that in the case of a tangential defect we always have $c < 0$ while in the case of a normal defect, for both kinds of velocities, in the considered range of s, the constant c changes its sign at least once. The case $c < 0$ corresponds to the periodic solutions of equation (3.27), whereas in the case $c > 0$ the solution contains a term of the form $e^{t\sqrt{c}}$ which is divergent as $t \to \infty$. Thus, in the case and only in the case of normal defects there exist elastic media in which the motion of the considered concentrated defect is divergent.

For $c < 0$ the general solution of equation (3.27) can be written in the form (we prefer to put down the corresponding position vector)

$$\zeta(t) = (\zeta_0 - c\dot{v}_0) + (v_0 - c\ddot{v}_0)t + c(\sqrt{-c})\ddot{v}_0 \sin t\sqrt{-c} + c\dot{v}_0 \cos t\sqrt{-c} \tag{3.28}$$

where $\zeta_0, v_0, \dot{v}_0, \ddot{v}_0$ are the initial values. Thus, the free motion of the considered defect is a uniform motion with oscillations of a very small amplitude and a very large frequency superposed on it; as $c \to 0$ we recover the Newton second law. The above phenomenon of oscillations clearly resembles the well-known "Zitterbewegung" of an electron.

We end this section with a remark concerning the order of the derived equations. It may seem strange that the order of the derived equations of motion is four, while in electrodynamics the corresponding equations are of second degree (of the fifth and third degree, respectively, if the radiation damping is taken into account; see Section 4). The difference is due to the different models of "particles"; in fact, the electron constitutes a counterpart of the simple layer surface potential, whereas the defect we consider is a double layer (see equation 2.7), a discontinuity in displacement being equivalent to a distribution of a double force. The equations of motion of a dipole electromagnetic charge can be shown to be of fourth (fifth) degree and, conversely, the equations for a surface distribution of forces in elasticity are of second (third) degree.

4. RADIATION DAMPING

It is evident that all processes described by the equation of motion (3.12) or (3.16) derived from a variational principle (3.1) are of reversible character. It can easily be shown that an irreversible term in the equations of motion of the form

$$s^{[ip]}\dddot{\zeta}_p \tag{4.1}$$

can be deduced from the Lagrangian density

$$s^{pq}\dot{v}_p\ddot{v}_q. \tag{4.2}$$

The symmetric part of s^{pq} yields in (4.2) a time derivative, which leads to no term in the equation of motion. It was noted before, however, (Section 2) that an introduction of the tensor $\overset{\text{ret}}{\mathbf{G}}$ instead of $\overset{+}{\mathbf{G}}$ which could perhaps in a natural way account for the irreversibility of the process, leads to the result $s^{pq} = 0$. Of course we may always add formally to the Lagrangian density a term of the form (4.2) with an indefinite tensor s^{pq}, there seems however to exist no justification for this procedure.

On the other hand it is obvious that during its motion a defect in an elastic field radiates elastic energy resulting in a damping which by analogy to electrodynamics will be called the radiation damping. This phenomenon leads to a new term in the equation of motion called the force due to the radiation damping.

In non-relativistic electrodynamics this force has the value $\frac{2}{3}(e^2/c^3)\dddot{\zeta}$ [cf. e.g. 7], well confirmed by experiment. This expression can be derived either by energy considerations or by the Wheeler–Feynman method [8]; in the case of the elastic field, the first method, mainly due to the presence of the term with the Heaviside function in the Green tensor, leads to very complicated calculations, we shall therefore make an attempt to adapt the Wheeler–Feynman procedure to defects in elastic field. It should be emphasized, however, that it has been verified that the energy method leads to an expression of exactly the same form, we may therefore hope that future calculations will prove that the coefficients (the tensor s^{pq}, see below) are also the same, as is the case in electrodynamics.

Consider therefore a system of interacting defects; the action of a defect β on defect α ($\alpha, \beta = 1, 2, \ldots$) is described by the force of β on α, given by the expression [see 4].

$$\underset{\alpha\beta}{\overset{+}{\mathbf{f}}} = -\left(\frac{\partial}{\partial \underset{\alpha}{\zeta}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \underset{\alpha}{\dot{\zeta}}}\right) \underset{\alpha}{t_{(n)}} \underset{\beta}{\overset{+}{\mathbf{u}}}(\underset{\alpha}{\zeta}, t) \, . \, \underset{\alpha}{\mathbf{U}} \tag{4.3}$$

where

$$\underset{\beta}{\overset{+}{\mathbf{u}}} = \frac{1}{2}\left(\underset{\beta}{\overset{\text{ret}}{\mathbf{u}}} + \underset{\beta}{\overset{\text{adv}}{\mathbf{u}}}\right). \tag{4.4}$$

Assuming that there are no body forces, the total force on a defect α, including the self-force, has the form

$$\underset{\alpha}{\mathbf{f}_T} = \sum_{\beta} \underset{\alpha\beta}{\overset{+}{\mathbf{f}}} \tag{4.5}$$

or, substituting from (4.3)

$$\underset{\alpha}{\mathbf{f}_T} = -\left(\frac{\partial}{\partial \underset{\alpha}{\zeta}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \underset{\alpha}{\dot{\zeta}}}\right) \underset{\alpha}{\mathbf{U}} \, . \, \underset{\alpha}{t_{(n)}} \sum_{\beta} \underset{\beta}{\overset{+}{\mathbf{u}}}. \tag{4.6}$$

The sum appearing in this formula can be transformed as follows:

$$\begin{aligned} \sum_{\beta} \underset{\beta}{\overset{+}{\mathbf{u}}} &= \sum_{\beta \neq \alpha} \underset{\beta}{\overset{\text{ret}}{\mathbf{u}}} - \frac{1}{2}\sum_{\beta \neq \alpha}\left(\underset{\beta}{\overset{\text{ret}}{\mathbf{u}}} - \underset{\beta}{\overset{\text{adv}}{\mathbf{u}}}\right) + \underset{\alpha}{\overset{+}{\mathbf{u}}} \\ &= \sum_{\beta \neq \alpha} \underset{\beta}{\overset{\text{ret}}{\mathbf{u}}} + \underset{\alpha}{\overset{+}{\mathbf{u}}} + \frac{1}{2}\left(\underset{\alpha}{\overset{\text{ret}}{\mathbf{u}}} - \underset{\alpha}{\overset{\text{adv}}{\mathbf{u}}}\right) - \frac{1}{2}\sum_{\beta}\left(\underset{\beta}{\overset{\text{ret}}{\mathbf{u}}} - \underset{\beta}{\overset{\text{adv}}{\mathbf{u}}}\right) \end{aligned} \tag{4.7}$$

and, similarly, with obvious notations,

$$\underset{\alpha}{\mathbf{f}_T} = \sum_{\beta \neq \alpha} \underset{\alpha\beta}{\overset{\text{ret}}{\mathbf{f}}} + \underset{\alpha\alpha}{\overset{+}{\mathbf{f}}} + \frac{1}{2}\left(\underset{\alpha\alpha}{\overset{\text{ret}}{\mathbf{f}}} - \underset{\alpha\alpha}{\overset{\text{adv}}{\mathbf{f}}}\right) - \frac{1}{2}\sum_{\beta}\left(\underset{\alpha\beta}{\overset{\text{ret}}{\mathbf{f}}} - \underset{\alpha\beta}{\overset{\text{adv}}{\mathbf{f}}}\right). \tag{4.8}$$

The last term in (4.7) or (4.8) is a characteristic of the whole system of defects moving in elastic field. Observe now, that since the singularities of $\underset{\beta}{\overset{\text{ret}}{\mathbf{u}}}$ and $\underset{\beta}{\overset{\text{adv}}{\mathbf{u}}}$ are the same, we have at all points of the medium, including the surfaces of the defects, the homogeneous Lamé equations

$$\mathbf{P} \, . \, \frac{1}{2}\sum_{\beta}\left(\underset{\beta}{\overset{\text{ret}}{\mathbf{u}}} - \underset{\beta}{\overset{\text{adv}}{\mathbf{u}}}\right) = 0. \tag{4.9}$$

We now make the following assumption constituting the essence of the Wheeler–Feynman method: both in the infinite past and the infinite future the total displacement field, i.e. the displacement field due to all defects, vanishes identically. Taking into account that

$$\overset{\text{ret}}{\underset{\beta}{\mathbf{u}}} \equiv 0 \quad \text{for } t = -\infty$$

$$\overset{\text{adv}}{\underset{\beta}{\mathbf{u}}} \equiv 0 \quad \text{for } t = +\infty$$

the above stated condition can be written in the form

$$\begin{aligned} \sum_{\beta}\left(\overset{\text{ret}}{\underset{\beta}{\mathbf{u}}} - \overset{\text{adv}}{\underset{\beta}{\mathbf{u}}}\right) &\equiv 0 \quad \text{for } t = -\infty \\ \sum_{\beta}\left(\overset{\text{ret}}{\underset{\beta}{\mathbf{u}}} - \overset{\text{adv}}{\underset{\beta}{\mathbf{u}}}\right) &\equiv 0 \quad \text{for } t = +\infty. \end{aligned} \tag{4.10}$$

Thus, denoting

$$\bar{\mathbf{u}}(\mathbf{x}, t) = \frac{1}{2}\sum_{\beta}\left[\overset{\text{ret}}{\underset{\beta}{\mathbf{u}}}(\mathbf{x}, t) - \overset{\text{adv}}{\underset{\beta}{\mathbf{u}}}(\mathbf{x}, t)\right]$$

we have the following initial-final problem for this function:

$$\begin{aligned} \mathbf{P}\bar{\mathbf{u}} &= 0 \quad \text{for any } t \\ \bar{\mathbf{u}}(\mathbf{x}, -\infty) &= \bar{\mathbf{u}}(\mathbf{x}, +\infty) = 0. \end{aligned} \tag{4.11}$$

Under certain regularity assumptions, just as in the case of the Cauchy problem, it can be proved that the above problem has only the trivial solution, i.e.

$$\bar{\mathbf{u}}(\mathbf{x}, t) = 0 \quad \text{for any } t \text{ and } \mathbf{x}. \tag{4.12}$$

Then, formulae (4.7) and (4.8) take the form

$$\begin{aligned} \sum_{\beta}\overset{+}{\underset{\beta}{\mathbf{u}}} &= \sum_{\beta\neq\alpha}\overset{\text{ret}}{\underset{\beta}{\mathbf{u}}} + \overset{+}{\underset{\alpha}{\mathbf{u}}} + \frac{1}{2}\left(\overset{\text{ret}}{\underset{\alpha}{\mathbf{u}}} - \overset{\text{adv}}{\underset{\alpha}{\mathbf{u}}}\right) \\ \underset{\alpha}{\mathbf{f}_T} &= \sum_{\beta\neq\alpha}\overset{\text{ret}}{\underset{\alpha\beta}{\mathbf{f}}} + \overset{+}{\underset{\alpha\alpha}{\mathbf{f}}} + \frac{1}{2}\left(\overset{\text{ret}}{\underset{\alpha\alpha}{\mathbf{f}}} - \overset{\text{adv}}{\underset{\alpha\alpha}{\mathbf{f}}}\right). \end{aligned} \tag{4.13}$$

In the last formula the first term in the right-hand side describes the action of other defects on the defect α, the second term constitutes the infinite self-force examined in Section 2 and finally the last term describes a (finite) additional contribution to the equations of motion; this is the required force due to the radiation damping, i.e.

$$\overset{\text{rad}}{\underset{\alpha}{\mathbf{f}}} = \frac{1}{2}\left(\overset{\text{ret}}{\underset{\alpha\alpha}{\mathbf{f}}} - \overset{\text{adv}}{\underset{\alpha\alpha}{\mathbf{f}}}\right). \tag{4.14}$$

It arose as a result of the assumption that due to the exchange of energy (interaction) between all defects in the field, initially and finally, all energy in the field is entirely absorbed and a complete rest occurs.

We now proceed to calculate this force in terms of the instantaneous characteristics of the defect. This can be done either directly or by expanding the considered functions into instantaneous potentials; we use here the first method. First, introduce the notations

$$\underset{\alpha}{\bar{\mathbf{u}}} = \frac{1}{2}\left(\underset{\alpha}{\overset{\text{ret}}{\mathbf{u}}} - \underset{\alpha}{\overset{\text{adv}}{\mathbf{u}}}\right), \qquad \bar{\mathbf{G}} = \frac{1}{2}\left(\overset{\text{ret}}{\mathbf{G}} - \overset{\text{adv}}{\mathbf{G}}\right) \tag{4.15}$$

and observe that

$$\bar{\mathbf{G}}(\zeta_\tau - \zeta_t, t-\tau) = \operatorname{sgn}(t-\tau)\overset{+}{\mathbf{G}}(\zeta_\tau - \zeta_t, t-\tau). \tag{4.16}$$

According to (2.9) we have the expression for the displacement $\underset{\alpha}{\bar{u}}^i(\underset{\alpha}{\zeta}, t)$:

$$\begin{aligned}\underset{\alpha}{\bar{u}}^i(\underset{\alpha}{\zeta}, t) = &-\mu \underset{\alpha}{U}_j \underset{\alpha}{n}_m \sigma^{jmpq} \int \mathrm{d}\tau \nabla_p \bar{G}^i_q(\underset{\alpha}{\zeta}_\tau - \underset{\alpha}{\zeta}_t, t-\tau) \\ &-\rho \underset{\alpha}{U}_j \underset{\alpha}{n}_m \delta^{jq} \int \mathrm{d}\tau \underset{\alpha}{\dot{\zeta}}{}^m_\tau \frac{\partial}{\partial \tau} \bar{G}^i_q .\end{aligned} \tag{4.17}$$

Hence, making use of (4.14) and (4.3) and dropping the irrelevant index α we have for the force to be calculated

$$\begin{aligned}\overset{\text{rad}}{f^i} = U_m U_l n_r n_t \Bigg[&\mu^2 \sigma^{mtkp} \sigma^{lrjs} \int \mathrm{d}\tau \nabla^i \nabla_p \nabla_s \bar{G}_{kj} \\ &+ \mu\rho\sigma^{mtkp}\delta^{jl} \times \int \mathrm{d}\tau \dot{\zeta}^r \nabla^i \nabla_p \frac{\partial}{\partial \tau} \bar{G}_{kj} + \mu\rho\sigma^{lrjs}\delta^{km}\dot{\zeta}^t \int \mathrm{d}\tau \nabla^i \nabla_s \frac{\partial}{\partial \tau} \bar{G}_{kj} \\ &+ \rho^2 \delta^{km} \delta^{jl} \dot{\zeta}^t \int \mathrm{d}\tau \dot{\zeta}^r \nabla^i \frac{\partial^2}{\partial \tau^2} \bar{G}_{kj} \\ &+ \mu\rho\sigma^{lrjs}\delta^{km}\delta^{it} \frac{\mathrm{d}t}{\mathrm{d}t} \times \int \mathrm{d}\tau \nabla_s \frac{\partial}{\partial \tau} \bar{G}_{kj} + \rho^2 \delta^{km} \delta^{jl} \delta^{it} \frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{d}\tau \dot{\zeta}^r \frac{\partial^2}{\partial \tau^2} \bar{G}_{kj} \Bigg].\end{aligned} \tag{4.18}$$

As in the whole paper we confine ourselves to expressions linear in the velocity of the defect or its derivatives. Thus, the fourth and the third terms in the right-hand side of (4.16) may be omitted, as easily verified by integration by parts. Integrating by parts in some of the remaining terms as well, we obtain

$$\begin{aligned}f^i = U_m U_l n_r n_t \rho^2 \Bigg[&c_2^4 \sigma^{mtkp} \sigma^{lrjs} \int \mathrm{d}\tau \nabla^i \nabla_p \nabla_s \bar{G}_{kj} \, c_2^2 \sigma^{mtkp} \int \mathrm{d}\tau \ddot{\zeta}^r \nabla^i \nabla_p \bar{G}^l_k \\ &- c_2^2 \sigma^{mtkp} \delta^{ri} \int \mathrm{d}\tau \ddot{\zeta}^j \nabla_j \nabla_p \bar{G}^1_k + \delta^{ri} \int \mathrm{d}\tau \dddot{\zeta}^t \bar{G}^{ml} \Bigg].\end{aligned} \tag{4.19}$$

The Green tensor entering the above formula has the form (cf. 2.19)

$$\begin{aligned}\bar{G}^{kj} &= \bar{G}^{kj}(\zeta_\tau - \zeta_t, t-\tau) = \bar{G}^{kj}(r_\tau, \theta) \\ &= -\operatorname{sgn}\theta \, . \, \frac{1}{4\pi\rho} \left\{ \int_{c_2}^{c_1} \left[\frac{2}{c^6} r^k r^j \delta'(\phi_c) - \frac{\delta^{kj}}{c^4} \delta(\phi_c) \right] \mathrm{d}c \right. \\ &\quad \left. + \frac{\delta^{kj}}{c_2^3} \delta(\phi_{c_2}) \right\}\end{aligned} \tag{4.20}$$

where

$$\phi_c = \theta^2 - \frac{r_\tau^2}{c^2} = \theta^2 \left[1 - \frac{1}{c^2}\left(v^2 + \theta v \dot{v} + \theta^2 \left(\frac{\dot{v}^2}{4} + \frac{v\ddot{v}}{3} \right) + \dots \right. \right] .$$

In our linear approximation we have the relations

$$\begin{aligned}\overset{(n)}{\zeta}{}^i_\alpha(\theta)\,\mathrm{sgn}\,\theta\,\overset{(m)}{\delta}\left(\theta^2-\frac{r^2_{\alpha\beta}}{c^2}\right)\Bigg|_{\beta=\alpha} &= \overset{(n)}{\zeta}{}^i_\alpha \lim_{r(t)\to 0}\mathrm{sgn}\,\theta\,\overset{(m)}{\delta}\left\{\theta^2-\frac{1}{c^2}[r^2(t)+\theta^2v^2+\ldots]\right\}\\ &= \overset{(n)}{\zeta}{}^i_\alpha \lim_{r(t)\to 0}\mathrm{sgn}\,\theta\,\overset{(m)}{\delta}\left(\theta^2-\frac{r^2(t)}{c^2}\right)\\ &= \overset{(n)}{\zeta}{}^i_\alpha\,\mathrm{sgn}\,\theta\,\overset{(m)}{\delta}(\theta^2)\end{aligned} \tag{4.21}$$

and the integrands of the integrals in (4.19) have the form

$$\begin{aligned}\dddot{\zeta}_t\bar{G}_{ml} &= -\frac{1}{12\pi\rho}(c_1^{-3}+2c_2^{-3})\delta_{ml}\dddot{\zeta}_t\,\mathrm{sgn}\,\theta\delta(\theta^2),\\ \ddot{\zeta}_r\nabla_i\nabla_p\bar{G}_{kl} &= \frac{1}{2\pi\rho}\{\tfrac{3}{5}(c_1^{-5}-c_2^{-5})\delta_{(kl}\,\delta_{p)i}+c_2^{-5}\delta_{kl}\delta_{pi}\}\ddot{\zeta}_r\,\mathrm{sgn}\,\theta\delta'(\theta^2),\\ \nabla_i\nabla_p\nabla_s\bar{G}_{jk} &= -\frac{1}{\pi\rho}\{\tfrac{1}{7}(c_1^{-7}-c_2^{-7})A_{kjspiu}+c_2^{-7}B_{kjspiu}\}r^u\,\mathrm{sgn}\,\theta\delta''(\theta^2),\end{aligned} \tag{4.22}$$

where

$$\begin{aligned}A^{iu}_{kjsp} &= 12\delta_{(kj}\delta^{(i}_s\delta^{u)}_{p)}+3\delta_{(kj}\delta_{s)p}\delta_{iu},\\ B_{kjspiu} &= 3\delta_{kj}\delta_{(sp}\delta_{i)u}.\end{aligned}$$

In the above integrals we replace integration with respect to τ by integration with respect to θ and we expand the quantities $\overset{(n)}{\zeta}{}^i_{(\tau)}$ into the Taylor series around the point $\tau = t$:

$$\begin{aligned}\overset{(n)}{\zeta}{}^i(\tau) = \overset{(n)}{\zeta}{}^i(t+0) &= \overset{(n)}{\zeta}{}^i+\overset{(n+1)}{\zeta}{}^i\theta+\overset{(n+2)}{\zeta}{}^i\frac{\theta^2}{2}+\overset{(n+3)}{\zeta}{}^i\frac{\theta^3}{6}\\ &+\overset{(n+4)}{\zeta}{}^i\frac{\theta^3}{24}+\overset{(n+5)}{\zeta}{}^i\frac{\theta^5}{120}+\ldots\end{aligned}$$

Further, making use of the identity

$$\mathrm{sgn}\,\theta\delta(\theta^2) = -\delta'(\theta)\dagger \tag{4.23}$$

we obtain

$$\begin{aligned}\mathrm{sgn}\,\theta\overset{(n)}{\delta}(\theta^2) &= \lim_{\varepsilon\to 0}\mathrm{sgn}\,\theta\overset{(n)}{\delta}(\theta^2-\varepsilon^2) = \lim_{\varepsilon\to 0}\mathrm{sgn}\,\theta\frac{\partial^{(n)}}{\partial(\theta^2)^n}\delta(\theta^2-\varepsilon^2)\\ &= \frac{\partial^{(n)}}{\partial(\theta^2)^n}\lim_{\varepsilon\to 0}\mathrm{sgn}\,\theta\delta(\theta^2-\varepsilon^2) = -\frac{\partial^{(n)}}{\partial(\theta^2)^n}\delta'(\theta).\end{aligned} \tag{4.24}$$

† It can be proved as follows:

$$\delta(\theta^2) = \lim_{a\to 0}\delta(\theta^2-a^2) = \lim_{a\to 0}\left[\frac{\delta(\theta-a)}{2a}+\frac{\delta(\theta+a)}{2a}\right]$$

Hence

$$\mathrm{sgn}\,\theta\delta(\theta^2) = \lim_{a\to 0}\left[\frac{\delta(\theta-a)}{2a}-\frac{\delta(\theta+a)}{2a}\right] = -\delta'(\theta)$$

In view of the above relations, bearing in mind the symmetry of the function $\overset{(n)}{\delta}(\theta^2)$ we finally obtain the expressions for the required integrals

$$
\begin{aligned}
\int d\theta \ddddot{\zeta}^i(t+\theta)\,\mathrm{sgn}\,\theta\delta(\theta^2) &= \int d\theta(\ddddot{\zeta}^i + \overset{.....}{\zeta}^i\theta)\,\mathrm{sgn}\,\theta\delta(\theta^2)\\
&= \int d\theta\overset{.....}{\zeta}^i\theta\,\mathrm{sgn}\,\theta\delta(\theta^2) = -\int d\theta\overset{.....}{\zeta}^i\theta\delta'(\theta) = \overset{.....}{\zeta}^i
\end{aligned}
$$

$$
\begin{aligned}
\int d\theta\ddot{\zeta}^i(t+\theta)\,\mathrm{sgn}\,\theta\delta'(\theta^2) &= \int d\theta\left(\ddot{\zeta}^i+\dddot{\zeta}^i\theta + \ddddot{\zeta}^i\frac{\theta^2}{2}+\overset{.....}{\zeta}^i\frac{\theta^3}{6}\right)\mathrm{sgn}\,\theta\delta'(\theta^2)\\
&= \int d\theta\left(\dddot{\zeta}^i\theta+\overset{.....}{\zeta}^i\frac{\theta^3}{6}\right)\mathrm{sgn}\,\theta\delta'(\theta^2) = -\int d\theta\left(\dddot{\zeta}^i\theta+\overset{.....}{\zeta}^i\frac{\theta^3}{6}\right)\frac{\partial}{\partial\theta^2}\delta'(\theta) \qquad (4.25)\\
&= -\int d\theta\delta(\theta)\frac{\partial}{\partial\theta^2}\left(\dddot{\zeta}^i+\overset{.....}{\zeta}^i\frac{\theta^2}{6}\right)\\
&= -\tfrac{1}{6}\overset{.....}{\zeta}^i
\end{aligned}
$$

$$
\begin{aligned}
\int d\theta[\zeta^i(t+\theta)-\zeta^i(t)]\,\mathrm{sgn}\,\theta\delta''(\theta^2) &= \int d\theta\left(\dot{\zeta}^i\theta+\ddot{\zeta}^i\frac{\theta^2}{2}+\dddot{\zeta}^i\frac{\theta^3}{6} + \ddddot{\zeta}^i\,\frac{\theta^4}{24}+ \overset{.....}{\zeta}^i\,\frac{\theta^5}{120}\right)\mathrm{sgn}\,\theta\delta''(\theta^2)\\
&= \int d\theta\left(\dot{\zeta}^i\theta+\dddot{\zeta}^i+\frac{\theta^3}{6} + \overset{.....}{\zeta}^i\,\frac{\theta^5}{120}\right)\mathrm{sgn}\,\theta\delta''(\theta^2)\\
&= -\int d\theta\left(\dot{\zeta}^i\theta+\dddot{\zeta}^i\frac{\theta^3}{6}+ \overset{.....}{\zeta}^i\,\frac{\theta^5}{120}\right)\frac{\partial^2}{\partial(\theta^2)^2}\delta'(\theta)\\
&= \int d\theta\delta(\theta)\frac{\partial^2}{\partial(\theta^2)^2}\left(\dot{\zeta}^i+\dddot{\zeta}^i\frac{\theta^2}{6}+\overset{.....}{\zeta}^i\,\frac{\theta^4}{120}\right) = \frac{\overset{.....}{\zeta}^i}{60}.
\end{aligned}
$$

After simple though cumbersome calculations, substituting into (4.17) we obtain the required force due to the radiation damping, namely we have

$$\overset{\mathrm{rad}}{f^i_\alpha} = s^{ip}\overset{.....}{\zeta}_p \qquad (4.26)$$

where

$$s_{ip} = \frac{\rho}{840\pi c_2^3}\{\delta_{ip}(s_1U^2+s_2U^2_{(n)})+s_3U_{(n)}n_{(i}U_{p)}+s_4U_iU_p+s_5U^2n_in_p\}. \qquad (4.27)$$

The coefficients of s_i are constants given by the simple formulae

$$
\begin{aligned}
s_1 &= 4s^7+10\\
s_2 &= 92s^7-112s^5+35s^3+6\\
s_3 &= -80s^7+140s^5-70s^3-18 \qquad (4.28)\\
s_4 &= 8s^7+6\\
s_5 &= 8s^7-28s^5+35s^3+34.
\end{aligned}
$$

Thus, the force due to the radiation damping is proportional to the fourth derivative of the velocity and hence it increases the order of the equation of motion by one, similarly to the equation of motion of an electron in Maxwell field.

Consider now some particular cases of this force, as in Section 3. Assume first that the defect moves in the direction of its normal, i.e. $v^i = vv^i$, $v^i = n^i$. We then have

(i) $U_{(n)} = 0$

$$\overset{\text{rad}}{f}{}^i_n = -\frac{\rho}{840\pi c_2^3} U^2 \ddddot{v} n^i (s_1+s_5) \tag{4.29}$$

(ii) $U^i = Un^i$

$$\overset{\text{rad}}{f}{}^i_\alpha = -\frac{\rho}{840\pi c_2^3} U^2 \dddot{v} (s_1+s_2+s_3+s_4+s_5). \tag{4.30}$$

Set now $v^i = vt^i$, $\mathbf{n}\,.\,\mathbf{t} = 0$; here we obtain

(i) $U_{(n)} = 0$

$$\overset{\text{rad}}{f}{}^i_\alpha = -\frac{\rho}{840\pi c_2^3} U^2 \dddot{v} t^i (s_1+s_4) \tag{4.31}$$

(ii) $U^i = Un^i$

$$\overset{\text{rad}}{f}{}^i_\alpha = -\frac{\rho}{840\pi c_2^3} U^2 \dddot{v} t^i (s_1+s_2); \tag{4.32}$$

in all cases it is readily verified that since

$$\begin{aligned} (s_1+s_2+s_3+s_4+s_5) &> 0 \\ (s_1+s_4) &> 0 \\ (s_1+s_2) &> 0 \\ (s_1+s_5) &> 0 \end{aligned} \tag{4.33}$$

for a positive $\dddot{v}$ we always have in the case $v^i = vn^i$

$$\overset{\text{rad}}{\mathbf{f}}_\alpha\,.\,\mathbf{n} < 0 \tag{4.34}$$

whereas in the case $v^i = vt^i$

$$\overset{\text{rad}}{\mathbf{f}}_\alpha\,.\,\mathbf{t} < 0 \tag{4.35}$$

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Résumé—Le mouvement d'un défaut concentré (point) dans un milieu élastique est examiné en se basant sur un principe de variations. Les équations de mouvement et le principe de la conservation de l'énergie sont déduites et examinées d'une façon assez détaillée. La localisation de l'expression de Lagrange permet de régulariser sa partie singulière et de déduire des équations différentielles de mouvement formelles. La force d'amortissement de radiation est introduite au moyen du processus Wheeler–Feynman. Dans l'exposé nous nous bornons à l'équation du second degré de Lagrange et par suite à des équations de mouvement du premier degré.

Zusammenfassung—Die Bewegung eines konzentrierten (Punkt-) Defektes in einem elastischen Medium wird auf Grund des Variationsprinzips untersucht. Die Bewegungsgleichung und das Energieprinzip werden abgeleitet und genau untersucht. Lokalisierung der Lagrange'schen Funktion ermöglicht es deren singulären Teil zu regeln und explizite Differentialgleichungen der Bewegung abzuleiten. Die Strahlungs-Dämpfungskräfte werden mittels des Wheeler–Feynman Vorganges eingeführt. In der Arbeit beschränken wir uns auf die quadratische Form der Lagrangeschen Funktion und somit auf die linearen Bewgungsgleichungen.

Абстракт—На основе вариационного принципа исследуется движение сосредоточного деффекта в упругой среде. Дан вывод уравнений движения и закона сохранения знергии, которые далее исследуются подробно. Локализация лагранжиана дает возможность регуляции их сингулярную частью и вывести точные дифференциальное уравнения движения. Приводится силу затухания излучения в смысле метода Чилера-Фейнмана. Автор ограничивается в зтой работе квадратным лагранжианом, и отсюда линейным уравнением движения.